

Nodal solutions to quasilinear elliptic equations on compact Riemannian manifolds

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ABSTRACT. We show the existence of nodal solutions to perturbed quasilinear elliptic equations with critical Sobolev exponent on compact Riemannian manifolds. A nonexistence result is also given.

1. Introduction

In this paper we investigate nodal solutions to quasilinear elliptic equations involving terms with critical growth on compact manifolds. Nodal solutions to scalar curvature type equation has been the subject of investigation by various authors. Among them, we cite D. Holcman[8], A. Jourdain[9], Z. Djadli and A. Jourdain[5]. This work is an extension to a previous one by Z. Djadli and A. Jourdain[5] where the authors studied the case of the Laplacian. We use variational methods based on the Mountain Pass Theorem as done in H. Brezis and L. Nirenberg [3] and some ideas due to H. Hebey regarding isometry concentration. We approach the problem via subcritical exponents, an idea originated by Yamabe. A non existence result of nodal solution based on a Pohozaev type identity is also given. Let (M, g) be a smooth compact Riemannian manifold $n \geq 3$, with or without boundary ∂M and $p \in (1, n)$. We use the notations of [5], let

$$W^{1,p}(M) = \begin{cases} H_1^p(M) & \text{if } \partial M = \emptyset \\ \overset{o}{H}_1^p(M) & \text{if } \partial M \neq \emptyset \end{cases}$$

where $H_1^p(M)$ is the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|_{1,p} = \|\nabla u\|_p + \|u\|_p$$

and $\overset{o}{H}_1^p(M)$ is the completion of $C_0^\infty(M)$ with respect to the same norm. Let G be a subgroup of the isometry group of (M, g) denoted $Isom(M)$. We assume that G is compact. We also consider τ an involutive isometry of

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(M, g) that is an element of $Isom(M)$ such that $\tau \circ \tau = id_M$. For x a point of M , we denote by $O_G(x)$ the orbit of x under the action of G . We say that G and τ commute weakly if for every $x \in M$, $\tau(O_G(x)) = O_G(\tau(x))$. We also say that the fixed points of τ splits M into two domains Ω_1 and Ω_2 stable under the action of G if

- (i) $M = \Omega_1 \cup \Omega_2 \cup F$, with $\Omega_1 \cap \Omega_2 = \emptyset$ and $mes(F) = 0$.
- (ii) $\tau(\Omega_1) = \Omega_2$, and $\forall \sigma \in G, \forall i = 1, 2 \quad \sigma(\Omega_i) = \Omega_i$

where F denotes the set of the fixed points of τ , that is $F = \{x \in M : \tau(x) = x\}$.

We say that a function $u \in W^{1,p}(M)$ is τ -antisymmetric if $u \circ \tau = -u$ a.e and G -invariant if for all $\sigma \in G$, $u \circ \sigma = u$ a.e. In what follows, we denote by $Card$ the cardinality of a set. We say that an operator L defined on $W^{1,p}(M)$ is coercive on a subspace X of $W^{1,p}(M)$ if there exists a positive real Λ such that for all $u \in X$,

$$\int_M L(u) u dv_g \geq \Lambda \|u\|_{1,p}^p.$$

Let a, f, h be smooth functions on M , and $p^* = \frac{np}{n-p}$, $q \in (p-1, p^*-1)$, we consider the following equation

$$(1) \quad \Delta_p u + a |u|^{p-2} u = f |u|^{p^*-2} u + h |u|^{q-1} u$$

with in case M has a boundary $u = 0$ on ∂M ; where $\Delta_p u = -div_g(|\nabla u|^{p-2} \nabla u)$. Under assumptions which will be precise later, we investigate nodal solutions of equation(1). By definition, a function $u \in W^{1,p}(M)$ is said to be a weak solution of the equation(1) if u satisfies (1) in the distribution sense.

We say that the functions a, f, h satisfy the conditions (C) at an interior point x_o of M if

$$\left\{ \begin{array}{ll} (i) & 1 < p < 2 \quad a(x_o) < 0 \\ (ii) & p = 2 \quad \frac{4(n-1)}{n-2} a(x_o) - Scal(x_o) + (n-4) \frac{\Delta f(x_o)}{f(x_o)} < 0 \\ (iii) & 2 < p < \frac{n}{2} \quad \frac{\Delta f(x_o)}{f(x_o)} < \frac{p}{n-3p+2} scal(x_o). \\ (iv) & \text{For all } 1 < p < n, \quad h(x_o) = 0 \quad \text{and } \Delta h(x_o) \leq 0. \end{array} \right.$$

We set $N = p^* - 1$ and let $q \in (p-1, N)$. Our main result in this paper reads as

THEOREM 1. *Let G be a compact subgroup of the isometry group of (M, g) , $n \geq 3$, τ an involutive isometry of (M, g) such that G and τ commute weakly. Let a, f and h be three smooth G -invariant and τ -invariant functions.*

We assume that:

- (1) *The operator $\varphi \rightarrow \Delta_p \varphi + a |\varphi|^{p-2} \varphi$ is coercive on the space $H = \{u \in W^{1,p}(M) : u \text{ is } G\text{-invariant and } \tau\text{-antisymmetric}\}$*
- (2) *f is positive on M and attains its maximum at an interior point x_o such that $\tau(O_G(x_o)) \cap O_G(x_o) = \emptyset$*
- (3) *The functions a, f, h satisfy the condition(C) at x_o .*

Then equation(1) possesses a nodal solution $u \in C^{1,\alpha}(M)$ which is G -invariant and τ -antisymmetric. Moreover, if we assume that the set F of fixed points of τ splits M into two domains Ω_1 and Ω_2 stable under the action of G , we can choose u such that the zero set of u is exactly $F \cup \partial M$.

2. A generic theorem of existence

First we give a regularity and a strong maximum results adapted to the context of manifolds from those of Tolksdorf [14], Guedda-Veron [7] and Vasquez [15] when dealing with Euclidian context. These results are also given by Druet [6] in the context of compact manifolds without boundary. The proofs are similar and based on Moser's iteration scheme.

THEOREM 2. ($C^{1,\alpha}$ -regularity) *Let (M, g) be a compact Riemannian n -manifold with or without boundary, $n \geq 2$, $p \in (1, n)$.*

If $u \in W^{1,p}(M)$ is a solution of equation (1) then $u \in C^{1,\alpha}(M)$.

PROOF. Put

$$(2) \quad g(x, u) = -a(x) |u|^{p-2} u + f(x) |u|^{p^*-2} u + h(x) |u|^{q-1} u$$

and

$$\tilde{h}(x) = \frac{g(x, u(x))}{1 + |u(x)|^{p-1}}.$$

Then

$$|\tilde{h}(x)| \leq \|a\|_\infty + \|f\|_\infty |u|^{p^*-p} + \|h\|_\infty |u|^{q+1-p}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Since $u \in W^{1,p}(M)$, we have $\tilde{h} \in L^{\frac{n}{p}}(M)$. The equation(1) reads as follows

$$(3) \quad \Delta_p u = \left(1 + |u(x)|^{p-1}\right) \tilde{h}.$$

Following arguments as in Guedda-Veron [7] and Vasquez [15] when dealing with Euclidian context we first show that any solution $u \in W^{1,p}(M)$ belongs to $L^q(M)$ for every $q \in [1, \infty[$. Let $k \geq 0$ and $v = \inf(|u|, C)$ where C is some positive constant.

Multiplying equation(3) by v^{kp+1} and integrating over M , we get

$$(4) \quad (kp+1) \int_M |u|^{kp} |\nabla u|^p dv_g = \int_M \text{sgn}(u) \tilde{h} \left(1 + |u(x)|^{p-1}\right) v^{kp+1} dv_g.$$

On other hand, we have

$$\left| \nabla |u|^{k+1} \right|^p = (k+1)^p |u|^{kp} |\nabla u|^p$$

so the equality (4) writes

$$(5) \quad \frac{kp+1}{(k+1)^p} \int_M \left| \nabla |u|^{k+1} \right|^p dv_g = \int_M \text{sgn}(u) \tilde{h} \left(1 + |u(x)|^{p-1}\right) v^{kp+1} dv_g.$$

Using Sobolev's inequality, we obtain for any fixed $\epsilon > 0$

$$\left\| |u|^{(k+1)} \right\|_{p^*}^p = \|u\|_{(k+1)p^*}^{(k+1)p}$$

$$\leq (K(n, p)^p + \epsilon) \left\| \nabla |u|^{k+1} \right\|_p^p + B \|u\|_{(k+1)p}^{(k+1)p}.$$

Taking into account the relation(5) and the Hölder's inequality we get

$$\begin{aligned} \|u\|_{(k+1)p^*}^{(k+1)p} &\leq (K(n, p)^p + \epsilon) \frac{(k+1)^p}{kp+1} C^{kp+1} \left(\|u\|_{(p-1)\frac{p^*}{p}}^{p-1} + Vol(M)^{1-\frac{p}{n}} \right) \left\| \tilde{h} \right\|_{\frac{n}{p}} \\ &\quad + B \|u\|_{(k+1)p}^{(k+1)p} \end{aligned}$$

where $K(n, p)$ is the best constant in the Sobolev's embedding $H_1^p(R^n) \subset L^{p^*}(M)$ (see T. Aubin [1]) and B is a positive constant depending on ϵ .

Now, taking $(k+1)p = p^*$ i.e. $k = \frac{p}{n-p}$, we obtain by the Hölder's inequality that

$$\begin{aligned} &\|u\|_{p^*(1+\frac{p}{n-p})} \leq \\ &\left\{ (K(n, p)^p + \epsilon) \frac{(k+1)^p}{kp+1} C^{kp+1} \left(Vol(M)^{\frac{1}{p^*}} + Vol(M)^{1-\frac{p}{n}} \right) \left\| \tilde{h} \right\|_{\frac{n}{p}} + B \right\}^{\frac{1}{p^*}} \\ &\quad \times \max(1, \|u\|_{p^*}). \end{aligned}$$

Consequently by a bootstrap arguments we get

$$u \in \bigcap_{1 \leq q < \infty} L^q(M).$$

Now using the Moser's iteration scheme we are going to show that $u \in L^\infty(M)$.

With the function g given as in (2), equation(1) reads

$$(6) \quad \Delta_p u = g.$$

For any $k > 1$, letting $t = k + p - 1$, we get

$$\left\| |u|^{\frac{t}{p}-1} \nabla u \right\|_p^p = \int_M |t|^{t-p} |\nabla u|^p dv_g.$$

and multiplying equation(6) by $|u|^k$ and integrating over M , we obtain

$$\begin{aligned} \int_M |u|^k \Delta_p u dv_g &= k \int_M |\nabla u|^p |u|^{k-2} u dv_g \\ (7) \quad &= \int_M g |u|^k dv_g. \end{aligned}$$

Using Sobolev's inequality, we obtain for any fixed $\epsilon > 0$

$$\begin{aligned} &\left\| |u|^{\frac{t}{p}-1} \nabla u \right\|_{p^*}^p = \|u\|_{t\frac{p^*}{p}}^t \\ &= (K(n, p)^p + \epsilon) \left(\frac{t}{p} \right) \left\| |u|^{\frac{t}{p}-1} \nabla u \right\|_p^p + B \|u\|_t^t. \end{aligned}$$

and since

$$\left\| |u|^{\frac{t}{p}-1} \nabla u \right\|_p^p = \int_M |t|^{t-p} |\nabla u|^p dv_g$$

and taking account of (7) we obtain

$$\begin{aligned} \left| \int_M |u|^k \Delta_p u dv_g \right| &= k \int_M |u|^{k-1} |\nabla u|^p dv_g = k \int_M |u|^{t-p} |\nabla u|^p dv_g \\ &= k \left\| |u|^{\frac{t}{p}-1} \nabla u \right\|_p^p \leq \|g\|_s \|u\|_{kr}^k \end{aligned}$$

where $r, s > 1$ are conjugate numbers. \square

Consequently

$$\|u\|_{t \frac{p^*}{p}}^t \leq (K(n, p)^p + \epsilon) \left(\frac{t}{p} \right)^p \|g\|_s \|u\|_{kr}^k + B \|u\|_t^t$$

and by Hölder's inequality we get

$$\begin{aligned} \|u\|_{t \frac{p^*}{p}}^t &\leq (K(n, p)^p + \epsilon) \left(\frac{t}{p} \right)^p \|g\|_s \|u\|_{rt}^k Vol(M)^{\frac{p-1}{rt}} \\ &\quad + B \|u\|_{rt}^t Vol(M)^{1-\frac{1}{r}}. \end{aligned}$$

Then

$$\begin{aligned} \|u\|_{t \frac{p^*}{p}} &\leq \left(\frac{t}{p} \right)^{\frac{p}{t}} Vol(M)^{\frac{p-1}{rt}} \max \{ (K(n, p)^p + \epsilon), B \} \\ &\quad \times \left(\|g\|_s + Vol(M)^{\frac{(r-1)t-p+1}{rt}} \right)^{\frac{1}{t}} \max(1, \|u\|_{rt}^t) \end{aligned}$$

or

$$(8) \quad \|u\|_{t \frac{p^*}{p}} \leq \left(\frac{t}{p} \right)^{\frac{p}{t}} A^{\frac{1}{t}} \max(1, \|u\|_{rt})$$

where A is a constant independent of t . Now we choose $r < \frac{p^*}{p} = \frac{n}{n-p}$ i.e. $s > \frac{n}{p}$ which is possible by the first part of the proof.

PROOF. Let $\alpha > 0$ such that $r(1 + \alpha) = \frac{p^*}{p}$ and $\beta = 1 + \alpha$. Let also $t = \beta^i$ where i is a positive integer; the recurrent relation(8) writes as

$$\|u\|_{r\beta^{i+1}} \leq \left(\frac{\beta^i}{p} \right)^{\frac{p}{\beta^i}} A^{\frac{1}{\beta^i}} \max(1, \|u\|_{r\beta^i})$$

and recurrently we get

$$\|u\|_{r\beta^{i+1}} \leq A^{\sum_{j=1}^i \frac{1}{\beta^j}} \frac{\beta^{p(\sum_{j=1}^i \frac{j}{\beta^j})}}{p(\sum_{j=1}^i \frac{1}{\beta^j})} \max(1, \|u\|_r).$$

Now since the series $\sum_{i=1}^{\infty} \frac{1}{\beta^i} = \frac{1}{\beta-1}$ and $\sum_{j=1}^{\infty} \frac{j}{\beta^j}$ are convergent, we get that $u \in L^\infty(M)$. At this stage the conclusion follows from theorem of P. Tolksdorf[14] \square

THEOREM 3. (*Strong maximum principle*) Let (M, g) be a compact Riemannian n -manifold with or without boundary, $p \in (1, n)$, and let $u \in C_o^1(M)$ be such that

$$\Delta_p u + f(., u) \geq 0 \quad \text{on } M,$$

f such that

$$\begin{cases} f(x, r) < f(x, s), & \forall x \in M \quad \forall 0 \leq r < s \\ |f(x, r)| \leq C \left(K + |r|^{p-2} \right) |r|, & \forall (x, r) \in M \times \mathbb{R} \end{cases}$$

where C and K are positive constants.

If $u \geq 0$ on M and u does not vanish identically, then $u > 0$ on $\text{int}(M) = M - \partial M$.

Let G be a compact subgroup of the isometry group of (M, g) and τ be an involutive isometry of (M, g) . We assume that G and τ commute weakly for some $x_1 \in M$, $\tau(O_G(x_1)) \cap O_G(x_1) = \emptyset$. Then

$$H = \{u \in W^{1,p}(M), u \text{ is } G\text{-invariant and } \tau\text{-antisymmetric}\}$$

is not trivial. Indeed H contains the test function given in section 3.

Denote by $\langle G, \tau \rangle$ the subgroup of the isometry group $\text{Isom}(M, g)$ generated by G and τ and by $K(n, p)$ the best constant in the Sobolev's embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{\frac{pn}{n-p}}(\mathbb{R}^n)$.

We consider the following functional J defined on H by

$$J(\varphi) = \int_M \left\{ \frac{1}{p} |\nabla \varphi|^p + a \frac{1}{p} |\varphi|^p - \frac{1}{p^*} f |\varphi|^{p^*} - \frac{1}{q+1} h |\varphi|^{q+1} \right\} dv_g.$$

In this section we establish the following generic theorem.

THEOREM 4. Let G be a compact subgroup of the isometry group of (M, g) , $n \geq 3$, τ an involutive isometry of (M, g) such that G and τ commute weakly and $\tau(O_G(x_1)) \cap O_G(x_1) = \emptyset$ for some $x_1 \in M$. Let also a , f and h be three smooth G -invariant and τ -invariant functions. We assume that f is positive on M and the operator $\varphi \rightarrow \Delta_p \varphi + a |\varphi|^{p-2} \varphi$ is coercive on H . We set $N = p^* - 1$ and let $q \in (p-1, N)$. We assume that for all x in M there exists $v \in H$, $v \neq 0$ such that

$$(8') \quad \sup_{t \geq 0} \{J(tv)\} < \frac{\text{Card} O_{\langle G, \tau \rangle}(x)}{K(n, p)^n f(x)^{\frac{n-p}{p}}}.$$

Then equation (1) possesses a nodal solution $u \in C^{1,\alpha}(M)$ which is G -invariant and τ -antisymmetric. Moreover, if we assume that the set F of fixed points of τ splits M into two domains Ω_1 and Ω_2 stable under the action of G , we can choose u such that the zero set of u is exactly $F \cup \partial M$.

2.1. The subcritical case. Now, following the strategy originated by Yamabe, we prove the existence of a nodal solution to the equation(1) for the subcritical exponent.

PROPOSITION 1. *Let G be a compact subgroup of the isometry group of (M, g) , $n \geq 3$, let τ be an involutive isometry of (M, g) such that G and τ commute weakly and such that for some $x_1 \in M$ $\tau(O_G(x_1)) \cap O_G(x_1) = \emptyset$. Let also a, f and h be three smooth G -invariant and τ -invariant functions. We assume that f is positive on M and that the operator $\varphi \rightarrow \Delta_p \varphi + a |\varphi|^{p-2} \varphi$ is coercive on H . We set $N = p^* - 1$, $q \in (p-1, N)$ and let ϵ_o be such that $0 < \epsilon_o \leq N - q$. Then for all ϵ such that $0 < \epsilon \leq \epsilon_o$ there exists $\varphi_\epsilon \in C^{1,\alpha}(M)$, G -invariant and τ -antisymmetric $\varphi_\epsilon \neq 0$ in M and $\varphi_\epsilon = 0$ on ∂M which is a nodal weak solution of the equation*

$$(9) \quad \Delta_p \varphi_\epsilon + a |\varphi_\epsilon|^{p-2} \varphi_\epsilon = f |\varphi_\epsilon|^{p^*-2-\epsilon} \varphi_\epsilon + h |\varphi_\epsilon|^{q-2} \varphi_\epsilon.$$

Moreover, if we assume that the set F of fixed points of τ splits M into two domains Ω_1 and Ω_2 stable under the action of G , we can choose φ_ϵ such that its zero set is exactly $F \cup \partial M$.

The proof of the Proposition(1) relies on the following Mountain-Pass Lemma of Ambrosetti and Rabinowitz([2])

LEMMA 1. *Let ϕ be a C^1 function on a Banach space X . Suppose that there exists a neighborhood U of 0 in X , a $v \in X \setminus U$ and a constant ρ such that $\phi(0) < \rho$, $\phi(v) < \rho$ and $\phi(u) \geq \rho$ for all $u \in \partial U$. Let Γ denote the class of continuous paths joining 0 to v and $c = \inf_{\gamma \in \Gamma} \max_{w \in \gamma} \phi(w)$.*

Then there is a sequence $(u_j)_j$ in X such that $\phi(u_j) \rightarrow c$ and $\phi'(u_j) \rightarrow 0$ in X^ (dual space of X).*

We recall the following concepts

DEFINITION 1. *Let X be a Banach and ϕ a function of class C^1 on X . We say that $u_n \in X$ is a Palais-Smale sequence at level c (Shortly a $(PS)_c$ sequence), if*

- (i) $\phi(u_n) \rightarrow c$
- (ii) $\phi'(u_n) \rightarrow 0$.

DEFINITION 2. *We say that ϕ satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence has a converging subsequence . We say that ϕ satisfies the (PS) condition if it satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.*

For any sufficiently small ϵ such that $0 < \epsilon \leq \epsilon_o$, we consider the C^1 -functional J_ϵ defined on the space H by

$$J_\epsilon(\varphi) = \int_M \left\{ \frac{1}{p} |\nabla \varphi|^p + \frac{1}{p} a |\varphi|^p - \frac{1}{p^* - \epsilon} f |\varphi|^{p^* - \epsilon} - \frac{1}{q + 1} h |\varphi|^{q+1} \right\} dv_g.$$

Following Brezis and Nirenberg ([3]), we show that the functional J_ϵ satisfies the assumptions of the Mountain-Pass Lemma, for every ϵ such that $0 < \epsilon \leq \epsilon_o$.

LEMMA 2. For every ϵ such that $0 < \epsilon \leq \epsilon_o$, there exists a ball U of radius independent of ϵ around 0 in H included in the unit ball, and a positive real number ρ independent of ϵ such that

- (i) $\forall \varphi \in U, J_\epsilon(\varphi) \geq \rho > 0$
- (ii) $\exists \psi \notin U$ such that $J_\epsilon(\psi) < \rho$.

PROOF. By the coercivity of the operator $\varphi \rightarrow \Delta_p \varphi + a |\varphi|^{p-2} \varphi$, there exists a positive real number Λ such that

$$J_\epsilon(\varphi) \geq \frac{\Lambda}{p} \|\varphi\|_{1,p}^p - \frac{1}{q+1} \|h\|_\infty \|\varphi\|_{q+1}^{q+1} - \frac{1}{p^* - \epsilon} \|f\|_\infty \|\varphi\|_{p^* - \epsilon}^{p^* - \epsilon}$$

and by the Sobolev's inequality, that is for every $\eta > 0$

$$\|\varphi\|_{q+1}^{q+1} \leq \left[(K(n,p)^p + \eta) \|\nabla \varphi\|_p^p + B \|\varphi\|_p^p \right]^{\frac{q+1}{p}},$$

one has

$$\begin{aligned} J_\epsilon(\varphi) \geq & \|\varphi\|_{1,p}^p \left[\frac{\Lambda}{p} - \frac{1}{q+1} \|h\|_\infty \max((K(n,p)^p + \eta), B)^{\frac{q+1}{p}} \|\varphi\|_{1,p}^{q+1-p} \right. \\ & \left. - \frac{1}{p^* - \epsilon} \|f\|_\infty \max((K(n,p)^p + \eta), B)^{\frac{p^* - \epsilon}{p}} \|\varphi\|_{1,p}^{p^* - p - \epsilon} \right] \end{aligned}$$

and since $q+1-p > 0$ and $p^* - p - \epsilon > 0$, there is a ball U included in the unit ball and a positive number ρ independent of ϵ with $0 < \epsilon \leq \epsilon_o$ such that for every $u \in \partial U$, $J_\epsilon(\varphi) \geq \rho$.

For $t > 0$,

$$\begin{aligned} J_\epsilon(t\varphi) \leq & t^p \left\{ \frac{1}{p} \|\nabla \varphi\|_p^p + \frac{1}{p} \|a\|_\infty \|\varphi\|_p^p - \frac{t^{q+1-p}}{q+1} \min_{x \in M} h(x) \|\varphi\|_{q+1}^{q+1} \right. \\ & \left. - \frac{t^{p^* - p - \epsilon}}{p^* - \epsilon} \min_{x \in M} f(x) \|\varphi\|_{p^* - \epsilon}^{p^* - \epsilon} \right\} \end{aligned}$$

so since $p^* - q - 1 > 0$, there is a sufficiently large t_o such that if $\psi = t_o \varphi$ then $\psi \notin U$ and $J_\epsilon(\psi) < \rho$ for ϵ sufficiently small. \square

Let P the class of continuous paths joining 0 to ψ and let $c_\epsilon = \inf_{\gamma \in P} \max_{w \in \gamma} J_\epsilon(w)$. Then by Lemma 2 there exists a $(PS)_{c_\epsilon}$ sequence in H .

Now we are going to show that each (PS) sequence satisfies the Palais-Smale condition.

LEMMA 3. Each Palais-Smale sequence for the functional J_ϵ is bounded.

PROOF. We argue by contradiction. Suppose that there exists a sequence $\{\varphi_j\}$ such that $J_\epsilon(\varphi_j)$ tends to a finite limit c , $J'_\epsilon(\varphi_j)$ goes to zero and φ_j to infinite in the $W^{1,p}(M)$ -norm. More explicitly we have for each $\psi \in W^{1,p}(M)$

$$\int_M \left\{ \frac{1}{p} |\nabla \varphi_j|^p + \frac{1}{p} a |\varphi_j|^p - \frac{1}{p^* - \epsilon} f |\varphi_j|^{p^* - \epsilon} - \frac{1}{q+1} h |\varphi_j|^{q+1} \right\} dv_g \rightarrow c$$

and

$$\begin{aligned} \int_M |\nabla \varphi_j|^{p-2} \nabla_i \varphi_j \nabla^i \psi dv_g + \int_M a |\varphi_j|^{p-2} \varphi_j \psi dv_g - \int_M f |\varphi_j|^{p^*-2-\epsilon} \varphi_j \psi dv_g \\ - \int_M h |\varphi_j|^{q-1} \varphi_j \psi dv_g \rightarrow 0 \end{aligned}$$

so for any $\eta > 0$ there exists a positive integer N such that for every $j \geq N$ one has

$$\left| \int_M \left\{ \frac{1}{p} |\nabla \varphi_j|^p + \frac{1}{p} a |\varphi_j|^p - \frac{1}{p^*-\epsilon} f |\varphi_j|^{p^*-\epsilon} - \frac{1}{q+1} h |\varphi_j|^{q+1} \right\} dv_g - c \right| \leq \eta$$

and

$$\begin{aligned} \left| \int_M |\nabla \varphi_j|^{p-2} \nabla_i \varphi_j \nabla^i \psi dv_g + \int_M a |\varphi_j|^{p-2} \varphi_j \psi dv_g - \int_M f |\varphi_j|^{p^*-2-\epsilon} \varphi_j \psi dv_g \right. \\ \left. - \int_M h |\varphi_j|^{q-1} \varphi_j \psi dv_g \right| \leq \eta \end{aligned}$$

In the particular case where $\psi = \varphi_j$, we get

$$\left| \int_M \left\{ \frac{1}{p} |\nabla \varphi_j|^p + \frac{1}{p} a |\varphi_j|^p - \frac{1}{p^*-\epsilon} f |\varphi_j|^{p^*-\epsilon} - \frac{1}{q+1} h |\varphi_j|^{q+1} \right\} dv_g - c \right| \leq \eta$$

and

$$\begin{aligned} \left| \int_M \left\{ |\nabla \varphi_j|^p dv_g + a |\varphi_j|^p dv_g - f |\varphi_j|^{p^*-\epsilon} dv_g \right. \right. \\ \left. \left. - \int_M h |\varphi_j|^{q+1} dv_g \right\} dv_g \right| \leq \eta. \end{aligned} \tag{10}$$

Then, we obtain

$$\begin{aligned} (11) \quad \left| \left(1 - \frac{p}{q+1}\right) \int_M \left(|\nabla \varphi_j|^p + a |\varphi_j|^p \right) dv_g + p \left(\frac{1}{q+1} - \frac{1}{p^*-\epsilon} \right) \int_M f |\varphi_j|^{p^*-\epsilon} dv_g - pc \right| \\ \leq \left(1 + \frac{1}{q+1}\right) p \eta \end{aligned}$$

and

$$\begin{aligned} (12) \quad \left| \left(1 - \frac{p}{p^*-\epsilon}\right) \int_M f |\varphi_j|^{p^*-\epsilon} dv_g + \left(1 - \frac{p}{q+1}\right) \int_M h |\varphi_j|^{q+1} dv_g - pc \right| \\ \leq (1+p) \eta. \end{aligned}$$

Now, since $f > 0$, there is a constant $C > 0$ such that

$$C \left(1 - \frac{p}{p^*-\epsilon}\right) \int_M |\varphi_j|^{p^*-\epsilon} dv_g \leq \left(1 - \frac{p}{q+1}\right) \|h\|_\infty \int_M |\varphi_j|^{q+1} dv_g + pc + (1+p) \eta$$

where $\|h\|_\infty = \max_{x \in M} |h(x)|$.

On the other hand since $p^* - \epsilon > q + 1$, for any $\nu > 0$, there exists a constant C'_ν such that $t^{q+1} \leq \nu t^{p^*-\epsilon} + C'_\nu$ for any $t \geq 0$. So

$$C(1 - \frac{p}{p^* - \epsilon}) \int_M |\varphi_j|^{p^*-\epsilon} dv_g \leq (1 - \frac{p}{q+1}) \|h\|_\infty \left(\nu \int_M |\varphi_j|^{p^*-\epsilon} dv_g + C_\nu \text{vol}(M) \right) + pc + (1+p)\eta$$

and

$$\left[C(1 - \frac{p}{p^* - \epsilon}) - (1 - \frac{p}{q+1}) \|h\|_\infty \nu \right] \int_M |\varphi_j|^{p^*-\epsilon} dv_g \leq cste.$$

Choosing $\nu > 0$ small enough so that $C(1 - \frac{p}{p^*-\epsilon}) - (1 - \frac{p}{q+1}) \|h\|_\infty \nu > 0$ and get

$$(13) \quad \int_M |\varphi_j|^{p^*-\epsilon} dv_g \leq cste.$$

By Lemma 2, we can choose ρ to be an $W^{1,p}(M)$ - norm such that

$$\inf_{\|\varphi\|_{1,p}=\rho} J_\epsilon(\varphi) > 0.$$

Letting $\psi_j = \rho \frac{\varphi_j}{\|\varphi_j\|_{1,p}}$, we obtain from (13) that

$$(14) \quad \int_M |\psi_j|^{p^*-\epsilon} dv_g = O \left(\frac{\rho^{p^*-\epsilon}}{\|\varphi_j\|_{1,p}^{p^*-\epsilon}} \right)$$

and by (11), we get

$$(15) \quad \left| \left(1 - \frac{p}{q+1} \right) \int_M (|\nabla \psi_j|^p + a |\psi_j|^p) dv_g + p \left(\frac{1}{q+1} - \frac{1}{p^*-\epsilon} \right) \frac{\|\varphi_j\|_{1,p}^{p^*-\epsilon-p}}{\rho^{p^*-\epsilon}} \int_M f |\psi_j|^{p^*-\epsilon} dv_g - pc \right| \leq \left(1 + \frac{1}{q+1} \right) p\eta$$

Letting j go to infinity, we obtain that $J_\epsilon(\psi_j)$ tends to zero. And since $\|\psi_j\|_{1,p} = \rho$, we have

$$\inf_{\|\varphi\|_{1,p}=\rho} J_\epsilon(\varphi) \leq J_\epsilon(\psi_j)$$

so

$$\inf_{\|\varphi\|_{1,p}=\rho} J_\epsilon(\varphi) \leq 0$$

hence a contradiction. Then the sequence $\{\varphi_j\}$ is bounded in $W^{1,p}(M)$. Now since $q < p^* - 1$, the Sobolev injections are compact, so the Palais-Smale condition is satisfied. \square

Now we are in position to prove Proposition1.

PROOF. (of Proposition1) Let C be the set of paths γ joining 0 and ψ and $c_\epsilon = \inf_{\gamma \in C} \sup_{t \in [0,1]} \gamma(t)$. As a consequence of Lemma2, there exists a sequence $\{\varphi_j\} \subset H$ such that $J_\epsilon(\varphi_j) \rightarrow c_\epsilon$ and $J'_\epsilon(\varphi_j) \rightarrow 0$ strongly in H' . By Lemma3, we can extract a subsequence still denoted $\{\varphi_j\}$ such that

$$\begin{aligned} \varphi_j &\rightarrow \varphi_\epsilon \text{ weakly in } H \\ \varphi_j &\rightarrow \varphi_\epsilon \text{ strongly in } L^r \quad \text{for } r < p^* \\ \varphi_j &\rightarrow \varphi_\epsilon \text{ a.e. in } M. \end{aligned}$$

Clearly φ_ϵ is a weak solution of the equation

$$(16) \quad \Delta_p \varphi_\epsilon + a |\varphi_\epsilon|^{p-2} \varphi_\epsilon - h |\varphi_\epsilon|^{q-1} \varphi_\epsilon - f |\varphi_\epsilon|^{p^*-2-\epsilon} \varphi_\epsilon = 0 \quad \text{in } H.$$

Now, to use the results of regularity, we must show that φ_ϵ satisfies the equation(16) weakly in $W^{1,p}(M)$. So we consider $\psi \in W^{1,p}(M)$ and the Haar measure denoted by $d\sigma$ on the isometric group G . Set $\bar{\psi}(x) = \int_G \psi(\sigma(x)) d\sigma$ for all $x \in M$, then $\bar{\psi}$ is G -invariant and it follows by multiplying the equation(16) by $\bar{\psi}$ and integrating over M that

$$\int_M \left\{ |\nabla \varphi_\epsilon|^{p-2} \nabla_i \varphi_\epsilon \nabla^i \bar{\psi} + a |\varphi_\epsilon|^{p-2} \varphi_\epsilon \bar{\psi} - h |\varphi_\epsilon|^{q-1} \varphi_\epsilon \bar{\psi} - f |\varphi_\epsilon|^{p^*-2-\epsilon} \varphi_\epsilon \bar{\psi} \right\} = 0$$

but

$$\nabla \int_G \psi(\sigma(x)) d\sigma = \int_G \nabla \psi(\sigma(x)) d\sigma$$

then

$$\begin{aligned} &\int_M \left\{ |\nabla \varphi_\epsilon|^{p-2} \nabla_i \varphi_\epsilon \nabla^i \left(\int_G \psi(\sigma(x)) d\sigma \right) + a |\varphi_\epsilon|^{p-2} \varphi_\epsilon \left(\int_G \psi(\sigma(x)) d\sigma \right) \right. \\ &\quad \left. - h |\varphi_\epsilon|^{q-1} \varphi_\epsilon \left(\int_G \psi(\sigma(x)) d\sigma \right) - f |\varphi_\epsilon|^{p^*-2-\epsilon} \varphi_\epsilon \left(\int_G \psi(\sigma(x)) d\sigma \right) \right\} dv_g \\ &= \int_M \int_G \left\{ |\nabla \varphi_\epsilon|^{p-2} \nabla_i \varphi_\epsilon \nabla^i \psi(\sigma(x)) + a |\varphi_\epsilon|^{p-2} \varphi_\epsilon \psi(\sigma(x)) \right. \\ &\quad \left. - h |\varphi_\epsilon|^{q-1} \varphi_\epsilon \psi(\sigma(x)) - f |\varphi_\epsilon|^{p^*-2-\epsilon} \varphi_\epsilon \psi(\sigma(x)) \right\} d\sigma dv_g = 0. \end{aligned}$$

Now, by the Fubini's theorem, we get

$$\begin{aligned} &\int_G \int_M \left\{ |\nabla \varphi_\epsilon|^{p-2} \nabla_i \varphi_\epsilon \nabla^i \psi(\sigma(x)) + a |\varphi_\epsilon|^{p-2} \varphi_\epsilon \psi(\sigma(x)) \right. \\ &\quad \left. - h |\varphi_\epsilon|^{q-1} \varphi_\epsilon \psi(\sigma(x)) - f |\varphi_\epsilon|^{p^*-2-\epsilon} \varphi_\epsilon \psi(\sigma(x)) \right\} dv_g d\sigma = 0 \end{aligned}$$

and since the functions a, h, f and φ_ϵ are G -invariant, the integral over M does not depend on $\sigma \in G$. Then

$$\int_M \left\{ |\nabla \varphi_\epsilon|^{p-2} \nabla_i \varphi_\epsilon \nabla^i \psi + a |\varphi_\epsilon|^{p-2} \varphi_\epsilon \psi - h |\varphi_\epsilon|^{q-1} \varphi_\epsilon \psi - f |\varphi_\epsilon|^{p^*-2-\epsilon} \varphi_\epsilon \psi \right\} dv_g = 0$$

thus φ_ϵ is a weak solution of the equation(16) in $W^{1,p}(M)$.

By the regularity theorem (Theorem2), $\varphi_\epsilon \in C^{1,\alpha}(M) \cap W^{1,p}(M)$, consequently $\varphi_\epsilon|_{\partial M} = 0$.

Now we are going to construct by mean of φ_ϵ a nodal solution of the subcritical equation(16). The construction is the same as in ([5]). Define

$$\psi_\epsilon(x) = \begin{cases} |\varphi_\epsilon| & \text{in } \Omega_1 \\ -|\varphi_\epsilon| & \text{in } \Omega_2 \end{cases}.$$

Since the set F of fixed points of τ is negligible set, it follows that $\psi_\epsilon \in H$ and arguing as above, there is $t_o > 0$ such that $J_\epsilon(t_o\psi_\epsilon) < 0$ for any $\epsilon \leq \epsilon_o$. Denote by C' the set of continuous paths joining 0 to $\psi_{o,\epsilon} = t_o\psi_\epsilon$ and let

$$c'_\epsilon = \inf_{\gamma \in C'} \max_{t \in [0,1]} J_\epsilon(\gamma(t)).$$

For any positive integer m , there exists a path $\tilde{\gamma}_m \in C$ such that

$$\max_{s \in [0,1]} J_\epsilon(\tilde{\gamma}_m(s)) \leq c_\epsilon + \frac{1}{m}.$$

We let as in ([5])

$$\gamma_m(s) = \begin{cases} -|\tilde{\gamma}_m(s)| & \text{in } \Omega_1 \\ |\tilde{\gamma}_m(s)| & \text{in } \Omega_2 \end{cases}$$

Clearly γ_m is a continuous path in the space H and let $s_m \in [0,1]$ such that

$$\begin{aligned} J_\epsilon(\gamma_m(s_m)) &= \max_{s \in [0,1]} J_\epsilon(\gamma_m(s)) \\ &\leq c_\epsilon + \frac{1}{m}. \end{aligned}$$

Now by the deformation lemma, there exists a continuous map

$$\eta_m : [0,1] \times H \rightarrow H$$

such that

$$(i) \quad \eta_m(t, \gamma_m(s_m)) = \gamma_m(s_m) \quad \text{for all } t \in [0,1]$$

and

$$\gamma_m(s_m) \notin J_\epsilon^{-1}\left(\left[c_\epsilon - \frac{1}{m}, c_\epsilon + \frac{1}{m}\right]\right)$$

$$(ii) \quad 0 \leq J_\epsilon(\gamma_m(s_m)) - J_\epsilon(\eta_m(t, \gamma_m(s_m))) \leq \frac{1}{m} \quad \text{for all } t \in [0,1]$$

$$(iii) \quad \|\eta_m(t, \gamma_m(s_m)) - \gamma_m(s_m)\| \leq \frac{1}{m} \quad \text{for all } t \in [0,1].$$

$$(iv) \quad \text{If } J_\epsilon(\gamma_m(s_m)) \leq c_\epsilon + \frac{1}{m}$$

then according to the deformation lemma either

$$J_\epsilon(\eta_m(1, \gamma_m(s_m))) \leq c_\epsilon - \frac{1}{m}$$

or for some $t_m \in [0, 1]$

$$\|J'_\epsilon(\eta_m(t_m, \gamma_m(s_m)))\| \leq \frac{1}{m}.$$

Now since we have

$$J_\epsilon(\gamma_m(s_m)) \leq c_\epsilon + \frac{1}{m}$$

by (iv) we get

$$J_\epsilon(\eta_m(1, \gamma_m(s_m))) \leq c_\epsilon - \frac{1}{m}.$$

Consequently since the path $s \rightarrow \eta_m(1, \gamma_m(s))$ joins 0 to $\psi_{o,\epsilon}$, we obtain from the definition of c_ϵ that

$$J_\epsilon(\gamma_m(s_m)) \geq c_\epsilon.$$

So the first part of (iv) cannot occur and then for some $t_m \in [0, 1]$

$$\|J'_\epsilon(\eta_m(t, \gamma_m(s_m)))\| \leq \frac{1}{m}.$$

Resuming, there exists $t_m \in [0, 1]$ such that

$$c_\epsilon \leq J_\epsilon(\eta_m(t_m, \gamma_m(s_m))) \leq c_\epsilon + \frac{1}{m}$$

and letting

$$\varphi_m = \eta_m(t_m, \gamma_m(s_m))$$

we get a sequence of elements of H such that

$$J_\epsilon(\varphi_m) \rightarrow c_\epsilon \quad \text{and} \quad J'_\epsilon(\varphi_m) \rightarrow 0.$$

Then as in the beginning of the proof of the Proposition1, there is a subsequence of the sequence (φ_m) still denoted (φ_m) which converges strongly in $L^{p^*-\epsilon}(M)$ to a weak solution φ_ϵ of the subcritical equation. Now by (iii) $\gamma_m(s_m) \rightarrow \varphi_\epsilon$ strongly in $L^{p^*-\epsilon}(M)$ then the convergence is also pointwise almost everywhere in M . Therefore $\varphi_\epsilon \geq 0$ on Ω_1 and $\varphi_\epsilon \leq 0$ on Ω_2 . Choosing a constant B such that the function $h(x, r) = a(x)|r|^{p-1} - f(x)|r|^{p^*-1} - h(x)|r|^q + B|r|^{p-1} \geq 0$ on $M \times R$ where $|r| \leq \|\varphi_\epsilon\|_{L^\infty(M)}$ we obtain that $\Delta_p \varphi_\epsilon + B\varphi_\epsilon^{p-1} \geq 0$ in Ω_1 and by the strong maximum principle(Theorem3) we get that $\varphi_\epsilon > 0$ in Ω_1 , and also we have $\varphi_\epsilon < 0$ in Ω_2 . \square

2.2. The critical case. Now we are going to show that the critical equation(1) has a nodal solution. First we state

PROPOSITION 2. *Let G be a compact subgroup of the isometry group of (M, g) , $n \geq 3$, let τ be an involutive isometry of (M, g) such that G and τ commute weakly and such that for some $x_1 \in M$ $\tau(O_G(x_1)) \cap O_G(x_1) = \phi$. Let also a , f and h be three smooth G -invariant and τ -invariant functions. We assume that f is positive on M and that the operator $\varphi \rightarrow \Delta_p \varphi + a|\varphi|^{p-2} \varphi$ is coercive on H . We set $N = p^* - 1$ and $q \in (p-1, N)$. Assume that the sequence $(\varphi_\epsilon)_\epsilon$ of solutions of the subcritical equations(9) admits a*

subsequence which converges in $L^k(M)$, $k > 1$, to $\psi \neq 0$. Then there exists $\varphi \in C^{1,\alpha}(M)$, G -invariant and τ -antisymmetric in M and $\varphi = 0$ on ∂M which is a nodal weak solution of the critical equation

$$\Delta_p \varphi + a |\varphi|^{p-2} \varphi = f |\varphi|^{p^*-2} \varphi + h |\varphi|^{q-2} \varphi.$$

PROOF. We first show that the set $(\varphi_\epsilon)_\epsilon$, $\epsilon \leq \epsilon_o$ of solutions to the subcritical equation(9) is bounded in $W^{1,p}(M)$. Let J be the functional defined on the Sobolev space $W^{1,p}(M)$ by

$$J(\varphi) = \int_M \left\{ \frac{1}{p} |\nabla \varphi|^p + \frac{1}{p} a |\varphi|^p - \frac{1}{p^*} f |\varphi|^{p^*} - \frac{1}{q+1} h |\varphi|^{q+1} \right\} dv_g,$$

$c = \inf_{\gamma \in C} \max_{t \in [0,1]} J(\gamma(t))$, where C denotes the set of paths γ joining 0 and ψ where ψ is the function given by Lemma1. With the same notations as in the proof of Proposition1, we have

$$\begin{aligned} c_\epsilon &= J_\epsilon(\varphi_\epsilon) \leq J(\varphi_\epsilon) + \frac{1}{p^*} \int_M f \left| |\varphi_\epsilon|^{p^*-\epsilon} - |\varphi_\epsilon|^{p^*} \right| dv_g \\ &\leq \inf_{u \in C} \max_{t \in [0,1]} J(u(t)) + \frac{1}{p^*} \max_M f \int_M \left| |\varphi_\epsilon|^{p^*-\epsilon} - |\varphi_\epsilon|^{p^*} \right| dv_g. \end{aligned}$$

So, since $\varphi_\epsilon \in C$,

$$c_\epsilon \leq c + \frac{1}{p^*} \max_M f \max_{t \in [0,1]} \int_M t^{p^*-\epsilon} \left| |\psi|^{p^*-\epsilon} - t^\epsilon |\psi|^{p^*} \right| dv_g.$$

Then

$$\lim_{\epsilon \rightarrow 0^+} \sup c_\epsilon \leq c$$

and the set $(\varphi_\epsilon)_\epsilon$ is bounded. So there is a sequence $(\varphi_n)_n$ such that $J'(\varphi_n) = 0$ and $J(\varphi_n) \rightarrow c'$. By arguments as in the proof of Lemma3, it follows that the sequence $(\varphi_n)_n$ is bounded in $W^{1,p}(M)$ and we have

$$\begin{aligned} \varphi_n &\rightarrow \varphi \quad \text{weakly in } W^{1,p}(M) \\ \varphi_n &\rightarrow \varphi \quad \text{strongly in } L^r(M) \text{ for } r < p^* \\ \varphi_n &\rightarrow \varphi \quad \text{pointwise a.e. in } M. \end{aligned}$$

Consequently $|\varphi_n|^{p^*-2} \varphi_n \rightarrow |\varphi|^{p^*-2} \varphi$ pointwise a.e. in M , and the sequence $|\varphi_n|^{p^*-2} \varphi_n$ is bounded in $(L^{p^*})'$ then by a well known theorem $|\varphi_n|^{p^*-2} \varphi_n \rightarrow |\varphi|^{p^*-2} \varphi$ weakly in $(L^{p^*})'$. The same is also true for the sequence $(|\varphi_n|^{q-1} \varphi_n)_n$ in $(L^{q+1})'$ and φ is weak solution of the critical equation(1). The remaining of the proof is the same as in the second part of the proof of Proposition1. \square

To show that the sequence $(\varphi_\epsilon)_\epsilon$ of solutions of the subcritical equations(9) admits a subsequence which converges to $\varphi \neq 0$ in $L^k(M)$, $k > 1$, we state.

LEMMA 4. *Suppose that*

(i) every subsequence of a sequence $(u_\epsilon)_\epsilon$ in $W^{1,p}(M)$ which converges in $L^k(M)$, with $k > 1$, converges to 0 (ii) For all $x \in M$, we can find $\delta > 0$ such that

(17)

$$K(n,p)^p(f(x))^{\frac{p}{p^*}} \lim_{\epsilon \rightarrow 0^+} \sup \left(\int_{B_x(\delta) \cap (B_x(\delta) - \partial M)} f |u_\epsilon|^{p^* - \epsilon} dv(g) \right)^{\frac{p^* - p}{p^*}} < 1.$$

where $B_x(\delta)$ is the ball centred at x and of radius δ . Then for any $x \in M$ there is $\delta = \delta(x) > 0$ such that

$$\lim_{\epsilon \rightarrow 0} \sup \left(\int_{B_x(\delta) \cap (B_x(\delta) - \partial M)} f |u_\epsilon|^{p^* - \epsilon} dv(g) \right) = 0$$

PROOF. Assume by contradiction that there is a $x_o \in M$ such that for any $\delta > 0$, $\limsup_{\epsilon \rightarrow 0^+} \left(\int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} f |u_\epsilon|^{p^* - \epsilon} dv_g \right) > 0$. Using Hölder's inequality, we get

$$\int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} f |u_\epsilon|^{p^* - \epsilon} dv_g \leq C \left(\int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} |u_\epsilon|^{p^*} dv_g \right)^{1 - \frac{\epsilon}{p^*}}$$

where C is a constant independent of ϵ , and for any $s > 1$

$$\begin{aligned} \left(\int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} |u_\epsilon|^{p^*} dv_g \right)^{1 - \frac{\epsilon}{p^*}} &\leq \left(\int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} |u_\epsilon|^{\frac{n(s+p-1)}{ns-p}} dv_g \right)^{\frac{(p^* - \epsilon)(ns-p)}{p^*(n(s+p-1))}} \\ &\times \left(\int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} |u_\epsilon|^{\frac{n(s+p-1)}{n-p}} dv_g \right)^{\frac{(p^* - \epsilon)(n(p-1)+p)}{p^*(n(s+p-1))}}. \end{aligned}$$

Consequently

$$\lim_{\epsilon \rightarrow 0^+} \sup \int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} |u_\epsilon|^{\frac{n(s+p-1)}{ns-p}} dv_g > 0$$

a contradiction with the fact that any subsequence of the sequence $(u_\epsilon)_\epsilon$ which converges in $L^k(M)$, for $k > 1$, converges to 0. \square

Now we are in position to prove Theorem4.

PROOF. (Proof of Theorem4) We show that the condition(i) of Lemma4 does not occur under the condition(ii).

Suppose by absurd that the condition(i) holds then

$$\lim_{\epsilon \rightarrow 0^+} \int_M h |u_\epsilon|^{q+1} dv_g = 0.$$

According to Lemma4 for every $x \in M$, there is $\delta(x) > 0$ such that

$$\lim_{\epsilon \rightarrow 0^+} \sup \int_{B_x(\delta) \cap (B_x(\delta) - \partial M)} f |u_\epsilon|^{p^* - \epsilon} dv_g = 0.$$

Now, since M is compact, there exist $x_1, \dots, x_s \in M$ such that $M = \cup_{1 \leq i \leq s} B_{x_i}(\delta_i(x_i))$.

Consequently

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \sup c_\epsilon &= \lim_{\epsilon \rightarrow 0^+} \sup J_\epsilon(u_\epsilon) \\ &= \lim_{\epsilon \rightarrow 0^+} \sup \int_M \left(\frac{q+1-p}{p(q+1)} h |u_\epsilon|^{q+1} + \frac{p^*-p}{p^*} f |u_\epsilon|^{p^*-\epsilon} \right) dv_g = 0 \end{aligned}$$

which contradicts the fact that for any ϵ with $0 < \epsilon \leq \epsilon_o$, $c_\epsilon \geq \rho > 0$.

So there exists $x_o \in M$ such that for any small $\delta > 0$,

(18)

$$K(n, p)^p (f(x_o))^{\frac{p}{p^*}} \lim_{\epsilon \rightarrow 0^+} \sup \left(\int_{B_x(\delta) \cap (B_x(\delta) - \partial M)} f |u_\epsilon|^{p^*-\epsilon} dv_g \right)^{\frac{p^*-p}{p^*}} \geq 1.$$

This gives

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup c_\epsilon &\geq \\ \frac{p^*-p}{p^*} \lim_{\epsilon \rightarrow 0} \sup \int_{B_x(\delta) \cap (B_x(\delta) - \partial M)} f |u_\epsilon|^{p^*-\epsilon} dv_g &\geq \frac{p}{n} f(x_o)^{1-\frac{n}{p}} K(n, p)^{-n}. \end{aligned}$$

Now if $CardO_{\langle G, \tau \rangle}(x_o) = +\infty$ we let $C > 0$ be some given constant and we choose $\delta > 0$ such that

$$C \lim_{\epsilon \rightarrow 0^+} \sup \int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} f |u_\epsilon|^{p^*-\epsilon} dv_g \leq \lim_{\epsilon \rightarrow 0^+} \sup c_\epsilon.$$

Now by taking C such that

$$C > K(n, p)^n f(x_o)^{\frac{p}{p^*}} \lim_{\epsilon \rightarrow 0^+} \sup c_\epsilon$$

we have

$$K(n, p)^n f(x_o)^{\frac{p}{p^*}} \lim_{\epsilon \rightarrow 0^+} \sup \int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} f |u_\epsilon|^{p^*-\epsilon} dv_g < 1.$$

Which contradicts (18).

If $CardO_{\langle G, \tau \rangle}(x_o) < +\infty$ we choose $\delta > 0$ small enough such that

$$CardO_{\langle G, \tau \rangle}(x_o) \lim_{\epsilon \rightarrow 0^+} \sup \int_{B_{x_o}(\delta) \cap (B_{x_o}(\delta) - \partial M)} f |u_\epsilon|^{p^*-\epsilon} dv_g \leq \lim_{\epsilon \rightarrow 0^+} \sup c_\epsilon$$

and taking account of (18), we obtain

$$\lim_{\epsilon \rightarrow 0^+} \sup c_\epsilon \geq K(n, p)^{-n} f(x_o)^{1-\frac{n}{p}} CardO_{\langle G, \tau \rangle}(x_o)$$

and since by construction of the sequence $(c_\epsilon)_\epsilon$,

$$c \geq \lim_{\epsilon \rightarrow 0^+} \sup c_\epsilon$$

it follows that

$$c \geq K(n, p)^{-n} f(x_o)^{1-\frac{n}{p}} CardO_{\langle G, \tau \rangle}(x_o).$$

But this contradicts the assumption (8') of Theorem 4. \square

3. Test functions

Let x_o be a point at the interior of M such that $f(x_o) = \max_{x \in M} f(x)$ and $O_G(x_o) \cap \tau(O_G(x_o)) = \phi$. Let ψ_η be the radial function defined by

$$\psi_{x_o, \eta} = \begin{cases} f(x_o)^{\frac{p-n}{p^2}} \eta^{\frac{n-p}{p^2}} (\eta + r^{\frac{p}{p-1}})^{1-\frac{n}{p}} C(n, p) - \mu & \text{for } 0 < r \leq \delta \\ 0 & \text{for } r > \delta \end{cases}$$

where $C(n, p) = \left(n \left(\frac{n-p}{p-1} \right)^{p-1} \right)^{\frac{n-p}{p^2}}$, $\mu = f(x_o)^{\frac{p-n}{p^2}} \eta^{\frac{n-p}{p^2}} (\eta + \delta^{\frac{p}{p-1}})^{1-\frac{n}{p}} C(n, p)$ and δ, η are small positive numbers and r is the geodesic distance function to the point x_o . Suppose that $O_G(x_o) = \{x_1, \dots, x_n\}$ and denote

$$\bar{\psi}_{x_o, \eta} = \sum_{i=1}^m \left(\psi_{x_i, \eta} - \psi_{\tau(x_i), \eta} \right).$$

We choose δ sufficiently small so that

$$\text{supp}(\psi_{x_i, \eta}) \cap \text{supp}(\psi_{x_j, \eta}) = \phi \quad \text{if } i \neq j$$

and

$$\text{supp}(\psi_{x_i, \eta}) \cap \text{supp}(\psi_{\tau(x_j), \eta}) = \phi \quad \text{for any } i \neq j.$$

Clearly $\bar{\psi}_{x_o, \eta}$ is G -invariant and τ -antisymmetric.

At this stage, we are able to prove Theorem1

PROOF. (Proof of Theorem1) Theorem1 will be proven if the condition(17) of Lemma4 holds and a fortiori if

$$0 < c < \frac{p}{n} (\max_M f)^{1-\frac{n}{p}} K(n, p)^{-n} \text{Card} O_{\langle G, \tau \rangle}(x_o).$$

and by the definition of c it suffices to show that

$$I(t\bar{\psi}_{x_o, \eta}) < \frac{p}{n} (\max_M f)^{1-\frac{n}{p}} K(n, p)^{-n} \text{Card} O_{\langle G, \tau \rangle}(x_o).$$

Now since

$$I(\bar{\psi}_{x_o, \eta}) = \text{card} O_{\langle G, \tau \rangle} I(\psi_{x_o, \eta})$$

we have to show that

$$I(t\psi_{x_o, \eta}) < \frac{p}{n} (\max_M f)^{1-\frac{n}{p}} K(n, p)^{-n}.$$

Put for simplicity

$$\psi_{x_o, \eta} = \psi_\eta.$$

The goal here is to compute the expansion in η of $I(t\psi_\eta)$. Now, classical computations of $\int_M |\nabla \psi_\eta|^p dv_g$ give

$$\begin{aligned} \int_M |\nabla \psi_\eta|^p dv_g &= C(n, p)^p \left(\frac{n-p}{p-1} \right)^p f(x_o)^{1-\frac{n}{p}} \omega_{n-1} \frac{p-1}{p} \\ &\times \left[\int_0^\infty (1+t)^{-n} t^{n(1-\frac{1}{p})} dt - \eta^{2(1-\frac{1}{p})} \frac{\text{Scal}(x_o)}{6n} \int_0^\infty (1+t)^{-n} t^{(n+2)-(1-\frac{1}{p})} dt \right] \\ &+ o(\eta^{2(1-\frac{1}{p})}). \end{aligned}$$

We Use the following relations, for any real numbers p, q with $p > q + 1$

$$I_p^q = \int_0^\infty (1+t)^{-p} t^q dt = \frac{\Gamma(q+1)\Gamma(p-q-1)}{\Gamma(p)}$$

where Γ denotes the Euler function. Such relations fulfill

$$\begin{aligned} I_n^{(n+2)(1-\frac{1}{p})} &= \frac{\Gamma\left((n+2)(1-\frac{1}{p})+1\right) \Gamma\left(\frac{n-3p+2}{p}\right)}{\Gamma\left(n(1-\frac{1}{p})+1\right) \Gamma\left(\frac{n}{p}-1\right)} I_n^{n(1-\frac{1}{p})} \\ &= a(n, p) I_n^{n(1-\frac{1}{p})}. \end{aligned}$$

We write

$$\begin{aligned} \int_M |\nabla \psi_\eta|^p dv_g &= C(n, p)^p \left(\frac{n-p}{p-1}\right)^p f(x_o)^{1-\frac{n}{p}} \omega_{n-1} \frac{p-1}{p} \\ (19) \quad &\times \left(1 - \eta^{2(1-\frac{1}{p})} \frac{Scal(x_o)}{6n} a(n, p)\right) I_n^{n(1-\frac{1}{p})} + o(\eta^{2(1-\frac{1}{p})}). \end{aligned}$$

Now, we compute $\int_M a(x) \psi_\eta^p$, and get

$$\begin{aligned} \int_M a(x) \psi_\eta^p dv_g &= \eta^{p-1} C(n, p)^p a(x_o) f(x_o)^{1-\frac{n}{p}} \omega_{n-1} \int_0^{+\infty} (1+t^{\frac{p}{p-1}})^{p-n} t^{n-1} dt \\ &\quad + o(\eta^{2(1-\frac{1}{p})}) \\ &= \frac{p-1}{p} \eta^{p-1} C(n, p)^p a(x_o) f(x_o)^{1-\frac{n}{p}} \omega_{n-1} I_{n-p}^{n(1-\frac{1}{p})-1} + o(\eta^{2(1-\frac{1}{p})}). \end{aligned}$$

Taking into account the following equalities

$$I_{n-p}^{n(1-\frac{1}{p})-1} = \frac{\Gamma(n)\Gamma(\frac{n}{p}-p)}{n(1-\frac{1}{p})\Gamma(n-p)\Gamma(\frac{n}{p}-1)} I_n^{n(1-\frac{1}{p})} = b(n, p) I_n^{n(1-\frac{1}{p})}$$

we obtain

$$\begin{aligned} \int_M a(x) \psi_\eta^p dv_g &= \\ (20) \quad &\frac{p-1}{p} \eta^{p-1} C(n, p)^p a(x_o) f(x_o)^{1-\frac{n}{p}} \omega_{n-1} b(n, p) I_n^{n(1-\frac{1}{p})} \\ &\quad + o(\eta^{2(1-\frac{1}{p})}). \end{aligned}$$

Finally we compute $\int_M f \psi_\eta^{p*} dv_g$ and get

$$\begin{aligned} \int_M f(x) \psi_\eta^{p*}(x) dv_g &= C(n, p)^{p*} f(x_o)^{1-\frac{n}{p}} \omega_{n-1} \frac{p-1}{p} \left[I_n^{n(1-\frac{1}{p})-1} \right. \\ &\quad \left. - \eta^{2(1-\frac{1}{p})} \left(\frac{\Delta f(x_o)}{2n f(x_o)} + \frac{Scal(x_o)}{6n} \right) I_n^{(n+2)(1-\frac{1}{p})-1} + o\left(\eta^{2(1-\frac{1}{p})}\right) \right] \end{aligned}$$

and since

$$I_n^{(n+2)(1-\frac{1}{p})-1} = \frac{\Gamma((n+2)(1-\frac{1}{p}))\Gamma(\frac{n+2}{p}-2)}{\Gamma(n(1-\frac{1}{p}))\Gamma(\frac{n}{p})} I_n^{n(1-\frac{1}{p})-1}$$

$$= c(n, p) I_n^{n(1-\frac{1}{p})-1}$$

we can write

$$\begin{aligned} \int_M f(x) \psi_\eta^{p^*}(x) dv_g &= C(n, p)^{p^*} f(x_o)^{1-\frac{n}{p}} \omega_{n-1} \frac{p-1}{p} I_n^{n(1-\frac{1}{p})-1} \\ &\times \left[1 - \eta^{2(1-\frac{1}{p})} \left(\frac{\Delta f(x_o)}{2nf(x_o)} + \frac{Scal(x_o)}{6n} \right) c(n, p) + o\left(\eta^{2(1-\frac{1}{p})}\right) \right]. \end{aligned}$$

Now letting in mind the equality

$$I_n^{n(1-\frac{1}{p})-1} = \frac{n-p}{n(p-1)} I_n^{n(1-\frac{1}{p})}$$

we obtain

$$\begin{aligned} \int_M f(x) \psi_\eta^{p^*}(x) dv_g &= C(n, p)^{p^*} f(x_o)^{1-\frac{n}{p}} \omega_{n-1} \frac{n-p}{np} I_n^{n(1-\frac{1}{p})} \\ (21) \quad &\times \left[1 - \eta^{2(1-\frac{1}{p})} \left(\frac{\Delta f(x_o)}{2nf(x_o)} + \frac{Scal(x_o)}{6n} \right) c(n, p) + o\left(\eta^{2(1-\frac{1}{p})}\right) \right]. \end{aligned}$$

Also we have, for any real q such that $\frac{n(p-1)+2p}{n-p} < q+1 < p^*$

$$\begin{aligned} \int_M h(x) \psi_\eta^{q+1}(x) dv_g &= \frac{p-1}{p} C(n, p)^{q+1} \eta^{\frac{n-p}{p^2}(q+1)} f(x_o)^{\frac{p-n}{p^2}(q+1)} \omega_{n-1} \\ &\times \left[h(x_o) \eta^{(1-\frac{n}{p})(q+1)+n(1-\frac{1}{p})} I_{(\frac{n}{p}-1)(q+1)}^{(n-1)(1-\frac{1}{p})-\frac{1}{p}} - \right. \\ &\left. \left(\frac{\Delta h(x_o)}{2n} + \frac{h(x_o) Scal(x_o)}{6n} \right) \eta^{(1-\frac{n}{p})(q+1)+(n+2)(1-\frac{1}{p})} I_{(\frac{n}{p}-1)(q+1)}^{n(1-\frac{1}{p})+1} \right] \\ &+ \int_0^\delta (\eta + r^{\frac{p}{p-1}})^{(1-\frac{n}{p})(q+1)} r^{n+1} dr \cdot o(\eta^{\frac{n-p}{p^2}(q+1)}) + o((\eta + r^{\frac{p}{p-1}})^{\frac{n-p}{p}}) \\ &= \frac{p-1}{p} C(n, p)^{q+1} \eta^{\frac{n-p}{p^2}(q+1)+(1-\frac{n}{p})(q+1)+n(1-\frac{1}{p})} f(x_o)^{\frac{p-n}{p^2}(q+1)} \\ &\quad \times \omega_{n-1} I_{(\frac{n}{p}-1)(q+1)}^{(n-1)(1-\frac{1}{p})-\frac{1}{p}} [h(x_o) - \\ &\quad \left(\frac{\Delta h(x_o)}{2n} + \frac{h(x_o) Scal(x_o)}{6n} \right) \eta^{2(1-\frac{1}{p})} e(n, p, q)] \\ &+ \int_0^\delta (\eta + r^{\frac{p}{p-1}})^{(1-\frac{n}{p})(q+1)} r^{n+1} dr \cdot o(\eta^{\frac{n-p}{p^2}(q+1)}) + o((\eta + r^{\frac{p}{p-1}})^{\frac{n-p}{p}}) \end{aligned}$$

where $e(n, p, q)$ is a constant.

Since $q+1 < p^* = \frac{np}{n-p}$, we have

$$\begin{aligned} &\frac{n-p}{p^2}(q+1) + (1-\frac{n}{p})(q+1) + n(1-\frac{1}{p}) \\ &= \frac{np - (n-p)(q+1)}{p} (1-\frac{1}{p}) > 0. \end{aligned}$$

We recall that the function $u : x \in R^n \rightarrow C(n, p)(1 + |x|^{\frac{p}{p-1}})^{1-\frac{n}{p}}$ realizes the equality in the embedding $H_1^p(R^n) \subset L_p(R^n)$ that is

$$\int_{R^n} |\nabla u|^p dx = K(n, p)^{-\frac{n}{p}} \int_{R^n} |u|^{p^*} dx$$

So from (19), (20) and (21), we get

$$\begin{aligned} I(t\psi_\eta) &= \left(t^p - \frac{p}{p^*}t^{p^*}\right) C(n, p)^p f(x_o)^{1-\frac{n}{p}} \omega_{n-1} \left(\frac{n-p}{p-1}\right)^p \frac{p-1}{p} I_n^{n(1-\frac{1}{p})} \frac{p}{n} \\ &\quad + F(t, n, p, \eta) \end{aligned}$$

and taking account of

$$K(n, p)^{-n} = \left(\frac{n-p}{p-1}\right)^p \frac{p-1}{p} C(n, p)^p \omega_{n-1} I_n^{n(1-\frac{1}{p})}$$

we obtain that

$$I(t\psi_\eta) = \left(t^p - \frac{p}{p^*}t^{p^*}\right) \frac{p}{n} K(n, p)^{-n} f(x_o)^{1-\frac{n}{p}} + F(t, n, p, \eta) + H(t, n, p, \eta)$$

where

$$\begin{aligned} F(t, n, p, \eta) &= t^p a(n, p) \frac{p-1}{p} C(n, p)^p \left(\frac{n-p}{p-1}\right)^p f(x_o)^{1-\frac{n}{p}} \omega_{n-1} I_n^{n(1-\frac{1}{p})} \\ &\quad \times \left\{ -\frac{Scal(x_o)}{6n} \eta^{2(1-\frac{1}{p})} + \left(\frac{p-1}{n-p}\right)^p a(x_o) \frac{b(n, p)}{a(n, p)} \eta^{p-1} \right. \\ &\quad \left. + t^{p^*-p} \frac{n-p}{n} \left(\frac{\Delta f(x_o)}{2nf(x_o)} + \frac{Scal(x_o)}{6n}\right) \frac{c(n, p)}{a(n, p)} \eta^{2(1-\frac{1}{p})} \right. \\ &\quad \left. + o\left(\eta^{2(1-\frac{1}{p})}\right) + o\left(\eta^{p-1}\right) \right\} \end{aligned}$$

and

$$\begin{aligned} H(t, n, p, \eta) &= -(t, t^{q+1-p} C(n, p)^{q+1-p} \\ &\quad \times \eta^{\frac{n-p}{p^2}(q+1) + (1-\frac{n}{p})(q+1) + n(1-\frac{1}{p})} f(x_o)^{\frac{p-n}{p}(1-\frac{q+1}{p})} \frac{I_n^{(n-1)(1-\frac{1}{p})-\frac{1}{p}}}{I_n^{n(1-\frac{1}{p})}} \\ &\quad \times \left[h(x_o) - \left(\frac{\Delta h(x_o)}{2n} + \frac{h(x_o) Scal(x_o)}{6n}\right) \eta^{2(1-\frac{1}{p})} e(n, p, q) \right] \Big\}. \end{aligned}$$

Let $t_1 \in [0, 1]$ such that $I(t_1\psi_\eta) = \sup_{t \in [0, 1]} I(t\psi_\eta)$, (t_1 is necessarily > 0). Since the function $\varphi(t) = t^p - \frac{p}{p^*}t^{p^*}$ attains its maximum on the interval $[0, 1]$ at $t_o = 1$, we get

$$\sup_{t \in [0, 1]} I(t\psi_\eta) < K(n, p)^{-n} f(x_o)^{1-\frac{n}{p}} \frac{p}{n} \left(\frac{n}{n-p}\right)^{\frac{n}{p}}$$

provided that $F(t_1, n, p, \eta) + H(t_1, n, p, \eta) < 0$. The assumptions $h(x_o) = 0$ and $\Delta h(x_o) \leq 0$ give us

$$H(t_1, n, p, \eta) \leq 0.$$

It remains now to show that $F(t_1, n, p, \eta) < 0$.

1) In the case $1 < p < 2$, $F(t_1, n, p, \eta)$ is equivalent to

$$\begin{aligned} G(t_1, n, p, \eta) &= \\ &= t_1^p a(n, p) \frac{p-1}{p} C(n, p)^p f(x_o)^{1-\frac{n}{p}} \omega_{n-1} I_n^{n(1-\frac{1}{p})} a(x_o) \frac{b(n, p)}{a(n, p)} \eta^{p-1} \end{aligned}$$

so we must have $a(x_o) < 0$

2) In the case $p = 2$, we have

$$\begin{aligned} F(t_1, n, 2, \eta) &\leq t_1^2 \frac{a(n, 2)}{6n} C(n, 2) (n-2)^2 f(x_o)^{1-\frac{n}{2}} \omega_{n-1} I_n^{\frac{n}{2}} \\ &\quad \times \left[-Scal(x_o) + \frac{6n}{(n-2)^2} a(x_o) \frac{b(n, 2)}{a(n, 2)} \right. \\ &\quad \left. + \frac{n-2}{n} \left(3 \frac{\Delta f(x_o)}{f(x_o)} + Scal(x_o) \right) \frac{c(n, 2)}{a(n, 2)} \right] \eta \\ &= t_1^2 \frac{a(n, 2)}{6n} C(n, 2) (n-2)^2 f(x_o)^{1-\frac{n}{2}} \omega_{n-1} I_n^{\frac{n}{2}} \\ &\quad \times \left[-Scal(x_o) + \frac{24(n-1)}{(n+2)(n-2)} a(x_o) \right. \\ &\quad \left. + \frac{n-2}{n} \left(3 \frac{\Delta f(x_o)}{f(x_o)} + Scal(x_o) \right) \frac{(n-4)n}{(n+2)(n-2)} \right] \eta \end{aligned}$$

and then the following condition must be satisfied

$$\frac{4(n-1)}{n-2} a(x_o) - Scal(x_o) + (n-4) \frac{\Delta f(x_o)}{f(x_o)} < 0.$$

3) In the case $2 < p < \frac{n}{2}$, to get $F(n, p, \eta) < 0$, we have to assume that

$$\left(1 - \frac{n-p}{n} \frac{c(n, p)}{a(n, p)} \right) Scal(x_o) > \frac{3(n-p)}{n} \frac{\Delta f(x_o)}{f(x_o)} \frac{c(n, p)}{a(n, p)}$$

i.e.

$$\frac{\Delta f(x_o)}{f(x_o)} < \frac{p}{n-3p+2} Scal(x_o).$$

□

4. Nonexistence results

In this section we give, by mean of a Pohozaev type identity, a nonexistence result.

PROPOSITION 3. *Let $n \geq 3$ and Ω be a star-shaped smooth domain of R^n with respect to the origin. Let $p \in (1, n)$. Suppose that $a \geq 0$, $\partial_r a \geq 0$, $\partial_r f \leq 0$, $h \leq 0$, $\partial_r h \leq 0$ and at least one of these inequalities is strictly then the critical equation(1) has no nodal solution.*

PROOF. A Pohozaev type identity for the p-Laplacian due to Guedda and Veron [7] reads as

$$\begin{aligned} n \int_{\Omega} H(x, u) dx + \int_{\Omega} \langle x, \nabla_x H(x, u) \rangle dx + \left(1 - \frac{n}{p}\right) \int_{\Omega} u g(x, u) dx \\ = \left(1 - \frac{1}{p}\right) \int_{\partial\Omega} \langle x, \nu \rangle \left| \frac{\partial u}{\partial \nu} \right| d\sigma \end{aligned}$$

where

$$g(x, u) = -a(x) |u|^{p-2} u + f(x) |u|^{p^*-2} u + h(x) |u|^{q-1} u$$

and

$$H(x, u) = \int_0^u g(x, s) ds.$$

ν is the unit outer normal vector field to $\partial\Omega$. A direct computation leads to the identity

$$\begin{aligned} -p \int_{\Omega} a |u|^p dx + \frac{np - (q+1)(n-p)}{q+1} \int_{\Omega} h |u|^{q+1} dx \\ - \int_{\Omega} \langle x, \nabla_x a \rangle |u|^p dx + \frac{n-p}{n} \int_{\Omega} \langle x, \nabla_x f \rangle |u|^{p^*} dx \\ + \frac{p}{q+1} \int_{\Omega} \langle x, \nabla_x h \rangle |u|^{q+1} dx = (p-1) \int_{\partial\Omega} \langle x, \nu \rangle |\partial_{\nu} u| d\sigma \end{aligned}$$

and letting $r = |x|$, we get

$$\begin{aligned} -p \int_{\Omega} a |u|^p dx + \frac{np - (q+1)(n-p)}{q+1} \int_{\Omega} h |u|^{q+1} dx \\ - \int_{\Omega} r \partial_r h |u|^p dx + \frac{n-p}{n} \int_{\Omega} r \partial_r f |u|^{p^*} dx \\ + \frac{p}{q+1} \int_{\Omega} |u|^{q+1} r \partial_r h dx = (p-1) \int_{\partial\Omega} \langle x, \nu \rangle |\partial_{\nu} u| d\sigma \end{aligned}$$

and the proof of the Proposition follows. \square

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